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TRANSITIVE STEINER AND KIRKMAN TRIPLE SYSTEMS OF ORDER 27

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ABSTRACT. There are 71 Steiner triple systems of order 27 whose automorphism groups are point-transitive, and there are 248 transitive Kirkman triple systems of order 27. Computational methods used to find these designs are outlined. The designs and some of their properties are presented.

1. STEINER TRIPLE SYSTEMS AND THEIR GROUPS

A *Steiner triple system* of order v, briefly $\mathrm{STS}(v)$, is a pair $(V, \mathscr{B})$ where V is a set of v elements and $\mathscr{B}$ is a set of 3-element subsets of V, with the property that every 2-subset of V appears in exactly one subset of $\mathscr{B}$. Sets in $\mathscr{B}$ are *triples*. An *automorphism* of an $\mathrm{STS}(v)$ is a permutation on V that maps each triple in $\mathscr{B}$ to a triple of $\mathscr{B}$, and the *automorphism group* is the group of all automorphisms of the STS. The STS is *transitive* if the automorphism group acts transitively on V; it is *cyclic* when the group contains Z_v as a subgroup, and *abelian* if there is an abelian subgroup of the automorphism group that acts transitively on V.

Steiner triple systems with large automorphism groups, and in particular transitive STS, are studied in large part because they yield examples of Steiner triple systems with interesting "regularity." There is an extensive literature on cyclic STS [2], but for the closely related case of transitive STS, especially nonabelian STS, little is known.

Transitive STS of order 21 have been constructively enumerated [5]; there are seven cyclic designs and three other transitive ones. Subsequently, Tonchev [8] constructively enumerated the transitive STS(25); there are three over $Z_5 \times Z_5$, in addition to the twelve over Z_{25} known since the 1930s [1].

For order 27, there are five possible automorphism groups to be considered. Three are the abelian groups $Z_3 \times Z_3 \times Z_3$, $Z_9 \times Z_3$, and Z_{27}. In addition, there are two nonabelian groups [3]. The first of these has all group elements of order 3, while the second has Z_9 as a subgroup. Every transitive STS(27) has (at least) one of these contained in its automorphism group. Cyclic STS(27)'s have been generated previously [1], as have the "1-rotational" STS(27)'s [7].

In this paper, we exhibit all transitive STS(27)'s. There are eight cyclic ones [1]; we find that there are 71 transitive STS(27)'s in total. In addition to presenting these designs, we give an extensive computational analysis of them.

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We list numbers of subdesigns, group orders, and whether or not the design is resolvable.

A *parallel class* in an STS is a spanning set of pairwise disjoint triples. A *resolution* is a partition of the triples of the STS into parallel classes. A Steiner triple system along with a resolution of it is a *Kirkman triple system*, or KTS. An automorphism of the STS is also an automorphism of a Kirkman triple system that it supports, *provided* the automorphism preserves the parallel classes of the resolution. In general, a Steiner triple system that is transitive can be resolvable in many ways; some Kirkman triple systems so arising may be transitive, while others are not. For a transitive STS, a *transitive resolution* (under a group Γ) is a resolution whose parallel classes are preserved under the action of Γ. A *transitive* KTS is a Kirkman triple system whose automorphism group acts transitively on elements (and, of course, preserves the resolution).

We examine all resolutions of the transitive STS(27) that are preserved by one of the groups acting transitively on 27 elements; we establish that there are precisely 248 nonisomorphic transitive Kirkman triple systems of order 27. We exhibit a compact representation of each, along with its group order.

Janko and van Trung [4] have previously found all 661 "2-rotational" KTS(27)'s, each necessarily having group order divisible by 13. Only one of their designs is also transitive as a Kirkman triple system, and hence 247 of the designs we exhibit appear to be previously unpublished.

Before presenting the catalogues of designs, we outline the computational methods used. Let Γ be one of the five groups of order 27, presented as a transitive permutation group on V, a set of 27 elements. The action of Γ partitions the 3-subsets of V into orbits $T_1, \dots, T_t$, and partitions the pairs into orbits $P_1, \dots, P_p$. Now form a matrix $A = A_{23}(\Gamma)$, whose (i, j) entry is the number of times a fixed pair in orbit P_i appears in triples of orbit T_j. Let U be a 0, 1-solution to the matrix equation $AU = \underline{1}$. Then U is the characteristic vector of a Steiner triple system whose automorphism group contains Γ.

Solutions of this matrix equation are in one-to-one correspondence with distinct STS(27)'s whose automorphism group contains Γ. Hence all such STS(27)'s can be found by solving a binary knapsack problem. However, we are interested only in nonisomorphic solutions. Applying a permutation π in the normalizer of Γ in Sym_{27} (the symmetric group on 27 symbols) carries an STS to an isomorphic, but possibly distinct, STS. Using the normalizer of Γ to eliminate duplicates in the search for the solutions to the matrix equation substantially reduces the number of solutions to be examined. Finally, we find one representative of each isomorphism type, and its automorphism group, using the graph isomorphism program "nauty" of McKay [6].

Parallel classes are found as follows. Form a 117-vertex block nonintersection graph, having a vertex representing each block, and two vertices adjacent when the corresponding blocks are disjoint. A parallel class is a 9-clique in this graph. Each parallel class generates an orbit of parallel classes under the action of Γ. Such an orbit may have all parallel classes having no triples in common (*a type-1 class*), or two parallel classes of the orbit may share a triple (*a type-2 class*). A transitive resolution contains only type-1 classes. To check resolvability (not necessarily transitive), form a graph whose vertices are the parallel classes, and make two vertices adjacent if the corresponding parallel classes have no common triple. A resolution is a 13-clique in this graph. In a typical case, the graph has thousands of vertices (parallel classes), and so we did not find all resolutions, but verified the existence or nonexistence of one.

Finding transitive resolutions is an easier matter. One first eliminates all type-2 classes. Then the remaining parallel classes fall into orbits under Γ. We form a graph whose vertices are the orbits of parallel classes, where each vertex has a *weight* equal to the number of parallel classes in the orbit. Two vertices are adjacent if no parallel class in the orbit represented by one vertex shares a triple with a parallel class of the orbit represented by the second vertex. Then a transitive resolution (that is, a transitive Kirkman triple system) is a clique of *weight* 13 in this graph.

Finally, determining the nonisomorphic transitive Kirkman triple systems was again performed by nauty. The success of the approach here rests on the effective solution of two difficult problems: an integer knapsack problem, and a clique problem. These are the same problem in disguise; the approach used is a heuristic method that quickly prunes the number of cases to be considered.

To determine the solutions of the binary knapsack problem $AU = J$, we use an algorithm called SYNTH. At present, SYNTH is ideally suited for problems in which A has several thousand columns but fewer than 100 rows, and determines all possible solutions. In this recursive algorithm, a column X of A which has a 1 in the first row is selected. All rows in which X has a 1 are marked for deletion, and so are all columns of A that are not orthogonal to X (that is, those that have nonzero inner product with column X). A *synthem* is a pair of binary vectors indicating which rows are active (not marked for deletion), and which columns are active. Synthems are used to pass information about these "conceptual" differences recursively to the main function in SYNTH. The synthem passed is then used to prune the possible solution space. This pruning process conceptually yields submatrix A' of A (without actually incurring the overhead of carrying out the deletions physically), and the algorithm recursively determines all solutions to subproblem $A'U' = J'$. The above deletions are relative only to the subproblem indicated by the synthem. The algorithm proceeds to select the second column X of A with a 1 in A's first row, and the process continues until all 1's in row one of A have been considered.

2. Transitive STS of order 27

In this section, we exhibit all 71 transitive Steiner triple systems of order 27. We report nonisomorphic solutions for each of the five groups; some designs are represented over more than one of the groups, but we assign each isomorphism type of design a unique number that is used throughout. When a design has a presentation over more than one group, we give a presentation of it over each relevant group.

For each design, the number of subdesigns has been determined; none has a subdesign of order 7 or 13, and the only admissible order for a subdesign is 9. We report the number of subdesigns for each design under column "S."

It is an easy exercise to see that each design has a parallel class, since there is some orbit of nine triples in each solution. However, not all of the designs are resolvable. Under the column "Res," we report on resolvability of each design as follows. "N" indicates that the design has no resolution at all. "R" indicates that it is resolvable but has no transitive resolution under the action of the group. "T" indicates that it has a transitive resolution, and the accompanying number is the number of such resolutions under the action of the group. In the case of design 50, the design has a unique resolution, and this resolution is transitive.

We have also computed the number of parallel classes in each design; this ranges from 478 (design 44) to 17641 (design 1). Design 1 is of course the affine geometry; hence, the fact that it maximizes the number of parallel classes, resolutions, transitive resolutions, and subdesigns, comes as no surprise.

Finally, we have also computed cycle structures [2] for each design. Design 1 has a 2-transitive automorphism group, and hence its cycle structure must be the same for all pairs of elements (i.e., it is *uniform*). Remarkably, design 2 is also uniform and has the same cycle structures as design 1. None of the remaining 69 are uniform, although designs 3 and 6 have only two different cycle structures.

For compactness, we list the designs with element set $\{a, b, \dots, y, z, A\}$, listing a block $\{a, b, c\}$ as abc. We only list orbit representatives for each design. To obtain the full set of 117 blocks, one applies the group generated by the generators given in each case to find the orbits of the representative triples; their union is the block set of the design.

2.1. **The nonabelian group of exponent three.** We present here the results for the nonabelian group of order 27 having all group elements of order 3. We use the set of three generators:

$$bcaxutmpjkirngfwsAyovehzqdl$$
$$drhealpicfmjxAzqgsbwtnukoyv$$
$$ftoljgazyqspbcndewumkixrvAh$$

There are forty nonisomorphic Steiner triple systems carried by this group; we present them here in compact form:

No.	Order	S	Res	Orbit Representatives
1	303264	39	T 729	*abc ade afg ahu aix ajp akA alq amo ant arv asz awy*
2	11664	12	T 567	*abc ade afg ahk aix ajp alq amo ant arv awy*
3	486	3	R	*abc ade afh aix ajp alq amo ant arv asz awy*
4	432	3	T 15	*abc ade afg ahi akA alv amo ant awy*
5	162	3	T 27	*abc ade afg ahi akA alq amo ant arv asz awy*
6	162	3	R	*abc ade afh air ajp alq amo ant asz*
7	81	3	T 21	*abc ade afg ahk aij alq amo arv awy*
8	81	0	T 6	*abc ade afg ahi akm anw arv*
9	81	0	T 3	*abc ade afg ahj ain aky asz*
10	54	3	R	*abc ade afh aij alq amo arv asz awy*

11	54	3	R	*abc ade afh ain ajp amo arv asz awy*
12	54	0	T9	*abc ade afg ahi akn alv amo*
13	54	3	T5	*abc ade afg ahk aij alr awy*
14	54	3	R	*abc ade afh ain ajm arv asz*
15	54	3	T3	*abc ade afh aix ajv aln asz*
16	54	3	R	*abc ade afi ahl ajs akA awy*
17	27	0	T5	*abc ade afg ahi akm ant arv asz awy*
18	27	0	T5	*abc ade afg ahi akA alq amo ans awy*
19	27	0	R	*abc ade afh aim ajp alq ant arv awy*
20	27	0	T3	*abc ade afg ahi akl amo ans*
21	27	0	T2	*abc ade afg ahi akm ans awy*
22	27	0	T3	*abc ade afg ahi akn alq amz*
23	27	0	T1	*abc ade afg ahi akn alr asz*
24	27	0	T2	*abc ade afg ahj aiq akn asz*
25	27	0	T1	*abc ade afh aij alq amz awy*
26	27	3	R	*abc ade afh aij alr asz awy*
27	27	0	R	*abc ade afh aij alv amo awy*
28	27	0	T1	*abc ade afh aim ajp aln arv*
29	27	0	T1	*abc ade afh aim ajt alq awy*
30	27	0	R	*abc ade afh ain ajp amz awy*
31	27	0	R	*abc ade afh aiq ajw ant arv*
32	27	0	R	*abc ade afh air ajn alq amo*
33	27	0	T2	*abc ade afh air ajp alt amo*
34	27	0	T1	*abc ade afh aix ajm alt arv*
35	27	0	T1	*abc ade afh aix ajm alv ant*
36	27	0	T2	*abc adf ahm akl ans*
37	27	0	T3	*abc adf ahq aik amz*
38	27	0	T1	*abc adf aht aim akl*
39	27	0	T2	*abc adf ahv aim akn*
40	27	0	T1	*abc adf ahx akl amz*

2.2. **The nonabelian group of exponent nine.** In this subsection, we treat the cases for the nonabelian group having a subgroup isomorphic to Z_9, having generators:

$$dulefghijkamnopqrscAvwxyztb$$
$$bcaxAwtvzuygkfjeidhlosnrmqp$$

This group carries 18 nonisomorphic triple systems, as follows:

No.	Order	S	Res	Orbit Representatives
1	303264	39	T 39	*abc ady aeu afi agv anz aqw*
2	11664	12	T 9	*abn adA aex afi agy*
3	486	3	R	*abf adA aex agy aqw*
41	81	0	T 5	*abl adw aeu afi agn*
42	27	0	R	*abl adf ags amA anz*
43	27	0	T 2	*abl adg aex amA anz*
44	27	0	T 1	*abl adi aex amz aqw*
45	27	0	R	*abl adi aex amA anz*
46	27	0	R	*abl adj ags amz aqw*
47	27	0	T 1	*abl adj ags amA anz*
48	27	0	N	*abl ads aeh amw anz*
49	27	0	T 1	*abl ads aei amw anz*
50	27	0	T 1	*abl adv aeh anz aqA*
51	27	0	T 2	*abl adv aei anz aqA*
52	27	0	T 3	*abl adv aez afi agw*
53	27	0	T 5	*abl adw aez afi agm*
54	27	0	T 4	*abl adx aeo afi agr*
55	27	0	T 4	*abl adx aeu afi agz*

2.3. **The group** $Z_3 \times Z_3 \times Z_3$. For the elementary abelian group Z_3^3, we use the three generators:

$$bcaefdhigkljnomqrptuswxvzAy$$
$$defghiabcmnopqrjklvwxyzAstu$$
$$jklmnopqrstuvwxyzAabcdefghi$$

This group carries five nonisomorphic triple systems, all of which are carried as well by at least one of the groups already handled.

No.	Order	S	Res	Orbit Representatives
1	303264	39	T 81	*abc adh ajt amA apx*
2	11664	12	T 99	*abc adh ajv akx alw*
3	486	3	T 81	*abc adg aei afj akw alA aoz*
6	162	3	T 81	*abc adh ajv akx alz*
41	81	0	T 29	*abc adg aej afm alv aoz aqx*

2.4. **The group** $Z_9 \times Z_3$. Here we give solutions for the abelian group $Z_9 \times Z_3$, with generators:

dlmefghijkanuopqrstbvwxyzAc
bcalnopqrstmduvwxyzAefghijk

This group carries 15 nonisomorphic triple systems, as follows:

No.	Order	S	Res	Orbit Representatives
1	303264	39	T 27	*abc adq ael afi agn aoy arv*
2	11664	12	T 9	*adv adt aep afi agq*
3	486	3	R	*abf adA aew agx aoy*
5	162	3	T 9	*abc adq ael afi agu aoy arv*
7	81	3	T 3	*abv adq ael afi agu*
8	81	0	T 3	*abm adr aep afi agy*
9	81	0	T 3	*abm adr aez afi agy*
56	54	3	R	*abf adt aep agA aoy*
57	54	3	R	*abf adt aez agA aoy*
58	27	0	T 2	*abm adg aes anx arv*
59	27	0	T 1	*abm adi aep aoy aqy*
60	27	0	T 2	*abm adi aes anx arv*
61	27	0	T 1	*abm adi aes any arv*
62	27	0	T 5	*abm adr aep afi ags*
63	27	0	T 5	*abm adr aez afi ags*

2.5. **The group** Z_{27}. The cyclic designs have been known for over fifty years [1]. We include them here for completeness, using the generator:

bcdefghijklmnopqrstuvwxyzAa

There are eight nonisomorphic cyclic STS(27) :

No.	Order	S	Res	Orbit Representatives
64	27	0	N	*abd ael afp ago ajs*
65	27	0	N	*abd ael afp agt ajs*
66	27	0	R	*abd ael afr ago ajs*
67	27	0	T 2	*abd ael afr agt ajs*
68	27	0	T 2	*abd aeo afl ahp ajs*
69	27	0	N	*abd aeo afl aht ajs*
70	27	0	N	*abd aer afv ahp ajs*
71	27	0	R	*abd aer afv aht ajs*

3. Transitive Kirkman triple systems

Forty-nine of the Steiner triple systems exhibited in §2 can be resolved in such a way that the Kirkman triple system is also transitive. In the appendix

(Supplement section at the end of this issue), we exhibit all nonisomorphic transitive Kirkman triple systems of order 27. There are 248 altogether. Hence we resort to a compact representation of the systems.

We list only those parallel classes that suffice to generate the thirteen parallel classes under the action of the groups (as given in §2). Each required parallel class is also listed in a compressed manner. Any automorphism of order 3 on the Kirkman system must either fix a parallel class, or must move the entire parallel class. If it fixes the parallel class, it may either fix the blocks of the parallel class, or permute them within the parallel class. In general, for each parallel class, some subgroup of the group of order 27 acting on the design fixes that parallel class. Hence it suffices to prescribe orbit representatives of the blocks in the parallel class in its stabilizing subgroup.

Each parallel class can therefore be succinctly described using the generators from §2 to specify the stabilizer of the parallel class, and listing orbit representatives for the parallel class under this subgroup. We specify the stabilizer of the parallel class by specifying a *subgroup code*, which is an integer from $\{0, \dots, 7\}$. It is interpreted as follows. For the two groups having all elements of order 3, let π_1, π_2, π_3 be the three generators (in the same order) as given in §2. For the two groups requiring two generators, let π_2, π_3 be the two generators as given in §2, and let $\pi_1 = \pi_2^3$. For Z_{27}, let π_3 be the generator given in §2, $\pi_2 = \pi_3^3$ and $\pi_1 = \pi_3^9$. It is *important* to remark that an automorphism of order 9 may move a parallel class, while the cube of the same automorphism fixes it; hence, we require these redundant generators in specifying the stabilizer of each parallel class.

Now with permutations chosen in this way, write the subgroup code as a 3-bit binary number $b_3b_2b_1$; the stabilizer is found by generating the minimal subgroup containing $\pi_1^{b_1}$, $\pi_2^{b_2}$, and $\pi_3^{b_3}$. Each parallel class is written as a subgroup code, followed by orbit representatives for the parallel classes; parallel classes are separated in the listing by colons.

Therefore, to recover the entire resolution, one first finds the stabilizer for each parallel class in turn, and applies all permutations in the stabilizer to reconstruct the parallel class. Then one applies the action of the entire group to find all thirteen parallel classes.

We give an example of the process here. Over the second nonabelian group, the following is a compact representation of a Kirkman triple system: $4\, adv\; egj\; fmz : 6\; anz : 2\; fnu$. There are three orbits of parallel classes. To recover the orbit representatives for all parallel classes, apply the second generator to adv, egj, fmz, both generators to anz, and the first generator to fnu. This yields the three parallel classes:

$$adv\ bxs\ chr\ \ egj\ tuA\ lop\ \ fmz\ kqw\ iny$$
$$anz\ dot\ epA\ \ bfq\ gru\ hsv\ \ ciw\ jlx\ kmy$$
$$fnu\ gov\ hpw\ \ iqx\ jry\ ksz\ \ act\ dlA\ bem$$

Now apply both generators to produce all parallel classes in the orbits of these three. The orbit lengths obtained are 9, 1, and 3 for the three parallel classes given:

adv	*bxs*	*chr*	*egj*	*tuA*	*lop*	*fmz*	*kqw*	*iny*
dew	*cuy*	*ils*	*fhk*	*bvA*	*mpq*	*gnt*	*arx*	*joz*
efx	*lvz*	*cjm*	*agi*	*buw*	*nqr*	*hoA*	*dsy*	*kpt*
fgy	*mtw*	*kln*	*dhj*	*uvx*	*ors*	*bip*	*cez*	*aqA*
ghz	*nxA*	*amo*	*eik*	*vwy*	*cps*	*jqu*	*flt*	*bdr*
hit	*boy*	*dnp*	*afj*	*wxz*	*clq*	*krv*	*gmA*	*esu*
ijA	*puz*	*eoq*	*dgk*	*txy*	*lmr*	*asw*	*bhn*	*cfv*
bjk	*qtv*	*fpr*	*aeh*	*yzA*	*mns*	*cdx*	*iou*	*glw*
aku	*rwA*	*gqs*	*dfi*	*btz*	*cno*	*ely*	*jpv*	*hmx*

anz	*dot*	*epA*	*bfq*	*gru*	*hsv*	*ciw*	*jlx*	*kmy*

fnu	*gov*	*hpw*	*iqx*	*jry*	*ksz*	*act*	*dlA*	*bem*
fow	*jst*	*env*	*irz*	*dmu*	*hqy*	*abl*	*gpx*	*ckA*
jnw	*hlu*	*fsA*	*dqz*	*kox*	*imv*	*bcg*	*ert*	*apy*

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BIBLIOGRAPHY

1. S. Bays, *Sur les systèmes cycliques de triples de Steiner différent pour n premier* (*ou puissance de nombre premier*) *de la forme* $6n+1$, Comment. Math. Helv. **4** (1932), 183–194.
2. M. J. Colbourn and R. A. Mathon, *On cyclic Steiner* 2*-designs*, Ann. Discrete Math. **7** (1980), 215–253.
3. M. Hall, Jr., *The theory of groups*, Macmillan, New York, 1959.
4. Z. Janko and T. van Trung, *On projective planes of order* 12 *with an automorphism of order* 13. *Part* 1: *Kirkman designs of order* 27, Geom. Dedicata **11** (1981), 257–284.
5. R. A. Mathon, K. T. Phelps, and A. Rosa, *A class of Steiner triple systems of order* 21 *and associated Kirkman systems*, Math. Comp. **37** (1981), 209–222.
6. B. D. McKay, *Practical graph isomorphism* (Proc. Tenth Manitoba Conf. Numerical Math. Computing), Congr. Numer. **30** (1981), 45–87.
7. K. Phelps and A. Rosa, *Steiner triple systems with rotational automorphisms*, Discrete Math. **33** (1981), 57–66.
8. V. D. Tonchev, *Transitive Steiner triple systems of order* 25, Discrete Math. **67** (1987), 211–214.

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